Relativized Succinct Arguments in the ROM Do Not Exist

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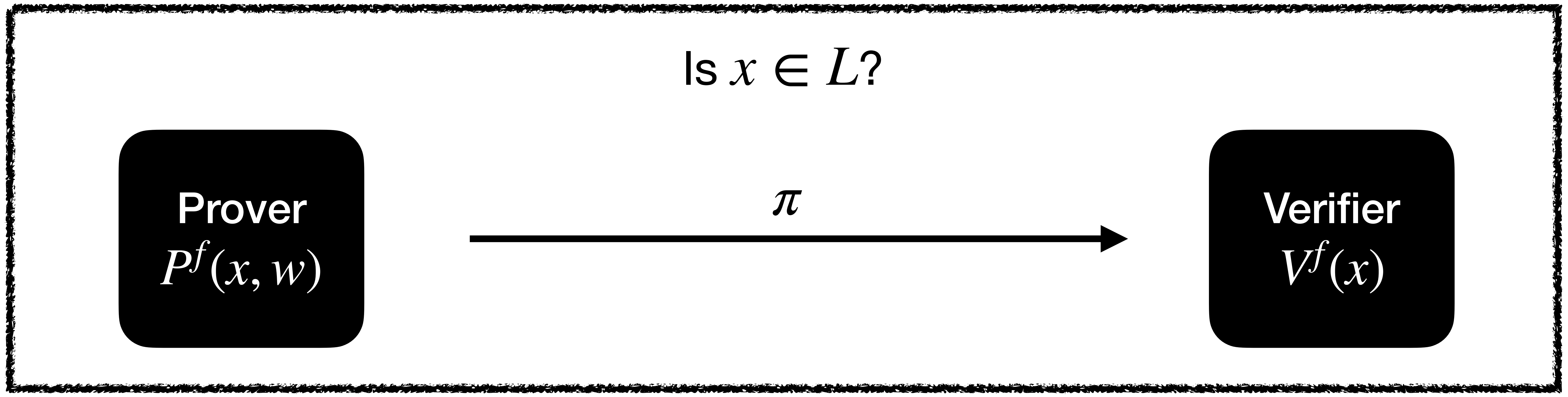
Succinct non-interactive arguments

SNARGs in the ROM



Random oracle $\mathcal{O} := \{\mathcal{O}_\ell\}_{\ell \in \mathbb{N}}$

uniform distribution over all functions $f\colon \{0,1\}^* \to \{0,1\}^\ell$

Completeness: $\forall$ instance-generating adversary A,

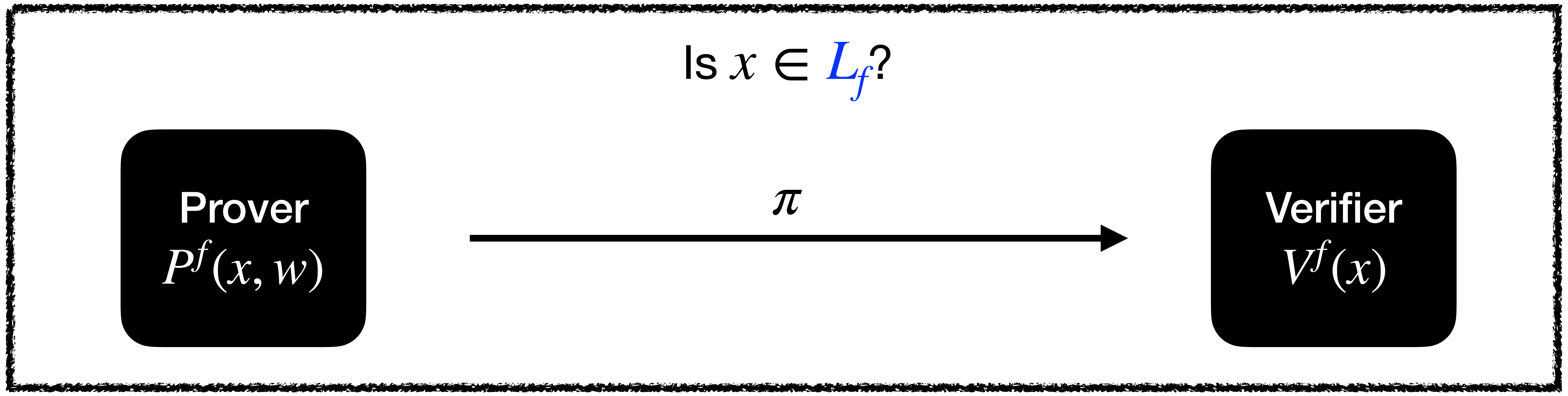
$$\Pr\left[x \in L \wedge V^f(x, \pi) = 1 \;\middle|\; \begin{array}{l} f \leftarrow \mathcal{O} \\ x \leftarrow A^f \\ \pi \leftarrow P^f(x) \end{array} \right] = 1.$$

Soundness: $\forall$ query-bounded and time-bounded adversary $\tilde{P}$,

$$\Pr\left[x \notin L \wedge V^f(x, \tilde{\pi}) = 1 \;\middle|\; \begin{array}{l} f \leftarrow \mathcal{O} \\ (x, \tilde{\pi}) \leftarrow \tilde{P}^f \end{array} \right] \le \epsilon.$$

What is a relativized argument in the ROM?

Relativization: The language L is relativized, $L = \{L_f : f \in \mathcal{O}\}$. e.g. $L_f := \{(x, y) : y = f(x)\}$



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uniform distribution over all functions f: $\{0,1\}^* \to \{0,1\}^\ell$

Completeness: $\forall$ instance-generating adversary A,

$$\Pr\left[x \in L_f \wedge V^f(x, \pi) = 1 \;\middle|\; \begin{array}{l} f \leftarrow \mathcal{O} \\ x \leftarrow A^f \\ \pi \leftarrow P^f(x) \end{array} \right] = 1.$$

Soundness: $\forall$ query-bounded and time-bounded adversary $\tilde{P}$,

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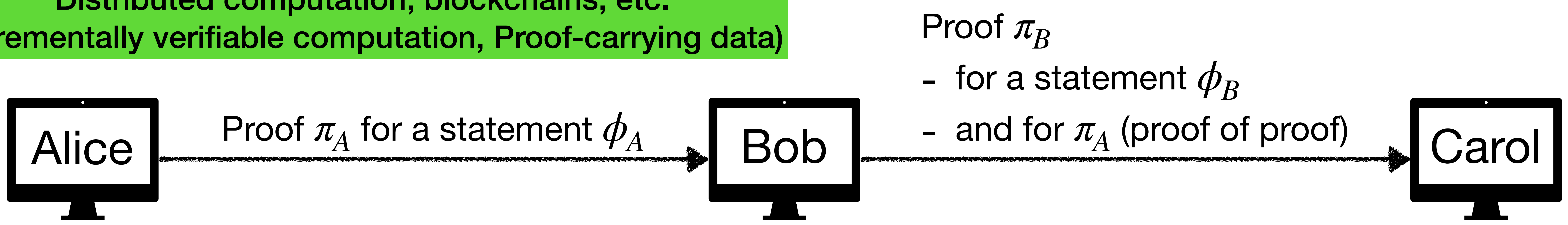
Why study relativized arguments? [1/2]

Motivation 1: Verifiable distributed computation

Distributed computation, blockchains, etc.
(Incrementally verifiable computation, Proof-carrying data)



$\mathrm{CSAT}_f := \{(C, x) : \exists w, C^f(x, w) = 1\}$

How does Bob produce π_B?

Let $\mathrm{ARG} = (P, V)$ be a SNARG for relativized CSAT:

Oracle recursive circuit $\mathscr{C}^f(\phi_B, (\phi_A, \pi_A))$
- **Check that ϕ_B is correct;**
- **Check that $V^f(\mathscr{C}, \phi_A, \pi_A) = 1$.**

$$\pi_B \leftarrow P^f(\mathscr{C}, \phi_B, (\phi_A, \pi_A))$$

Why study relativized arguments? [2/2]

Motivation 2: Efficiency

Recurring cryptographic computations show up a lot:

- Correctness proof of encryption/decryption, signature verification, **hash function**, etc.

hash function

e.g. $L_s := \{(n, y) \in \mathbb{N} \times \{0,1\}^{|s|} : \exists x \in \{0,1\}^s, H_s^{(n)}(x) = y\}$

Necessary to construct hash function with small size??

SNARGs for L_s are expensive (|circuit that iteratively applies H_s for n times| = $\Omega(n|H_s|)$).

If $\mathrm{NP}^{H_s} \subseteq \mathrm{ARG}^{H_s}$, SNARGs for L_s do not depend on $|H_s|$

Potential alternative route:

- Treat the hash function as an oracle.
- Relativized arguments do not depend on complexity of the hash functions. 🥳

More generally, relativization removes the need for optimizing the recurring sub-computation.

Do relativized SNARGs exist in oracle models? Yes!

Existing relativized SNARGs

Relativized SNARGs exist in some oracle models:
- Signed random oracle model (SROM) [CT10]
- Low-degree random oracle (LDROM) [CCS22]
- Arithmetized random oracle model (AROM) [CCGOS23]

Hard to instantiate!

How about the random oracle model?

Popular belief: No.
Popular intuition: Relativized PCPs/IOPs do not exist in the ROM [CL20].
Counterexample to popular belief:
- Relativized PCPs/IOPs do not exist in the LDROM [CL20].
- Relativized SNARGs exist in the LDROM [CCS22].

Our results

Relativized arguments in the random oracle model do not exist.

verifier query complexity to the RO

Trivial Baseline 1. $\mathsf{DTIME}^{\mathcal{O}}[\mathsf{t}] \subseteq \mathsf{ARG}^{\mathcal{O}}[\mathsf{vq} = t]$.

Verifier computes everything itself.

Theorem 1. $\mathsf{DTIME}^{\mathcal{O}}[\mathsf{t}] \not\subseteq \mathsf{ARG}^{\mathcal{O}}[\mathsf{vq} = o(t)]$.

argument proof size

Trivial Baseline 2. $\mathsf{NTIME}^{\mathcal{O}}[\mathsf{t}] \subseteq \mathsf{ARG}^{\mathcal{O}}[\mathsf{as} = t]$.

Prover sends the entire witness.

Theorem 2. $\mathsf{NTIME}^{\mathcal{O}}[\mathsf{t}] \not\subseteq \mathsf{ARG}^{\mathcal{O}}[\mathsf{as} = o(t)]$.

Existence of IVC/PCD in the ROM still remains open.

Corollary. Relativized IVC/PCD does not exist in the ROM!

Note.
- The results hold for SNARGs secure against query-bounded and time-bounded adversaries.
- Similar results hold for interactive arguments.

Separation between NTIME and ARG

Hard language in $\text{NTIME}^{\mathcal{O}}[t]$

Lemma.

argument proof size

There exists $L_{\mathcal{O}}$ such that $L_{\mathcal{O}} \in \text{NTIME}^{\mathcal{O}}[t]$ and $L_{\mathcal{O}} \notin \text{ARG}^{\mathcal{O}}[\text{as} = o(t)]$.

$$L_{\mathcal{O}} := \{L_f : f \in \mathcal{O}\}$$

$$L_f := \left\{ x \in \{0,1\}^n : \begin{array}{l} x = 0^n \\ \wedge \exists w \in \{0,1\}^{t(n)}, \forall i \in [t(n)], f(w\|i)_1 = 0 \end{array} \right\}$$

	$w\|1$	$w\|2$	$w\|3$	$w\|4$	$w\|5$	
f	1000	0101	1111	0000	0010	$x \notin L_f$
f	0001	0111	0110	0111	0110	$x \in L_f$

Why is L_f hard?

- Needs $t(n)$ queries to be sure that $x \in L_f$ or not.
- Flipping even one bit of f could change the membership of x.

	$w\|1$	$w\|2$	$w\|3$	$w\|4$	$w\|5$	
f	1000	0101	0100	0000	0010	$x \notin L_f$
f'	0000	0101	0100	0000	0010	$x \in L_f$

Proof outline

$$L_f := \left\{ x \in \{0,1\}^n : \begin{array}{l} x = 0^n \\ \wedge \exists w \in \{0,1\}^{t(n)}, \forall i \in [t(n)], f(w\|i)_1 = 0 \end{array} \right\}$$

1. Fix $x := 0^n$ for some n.
2. Consider $f^\star \in \mathcal{O}$ such that $x \notin L_{f^\star}$.
3. For every $w \in \{0,1\}^{t(n)}$, define f_w to be $f^\star$, except that $f_w(w\|i)_1 = 0$ for every $i \in [t(n)]$.
   - $f_w \in \mathcal{O}$.
   - $x \in L_{f_w}$.

	$w\|1$	$w\|2$	$w\|3$	$w\|4$	$w\|5$
$f^\star$	1001	0111	1110	0000	1010
f_w	0001	0111	0110	0000	0010

Intuition: without a long argument string, argument verifier cannot make meaningful queries!

4. **Claim***: For every $f \in \mathcal{O}$, there exists a large set $Q_f \subseteq \{0,1\}^{t(n)}$ such that

   $\forall w \in Q_f,\ \forall i \in [t(n)],\ \Pr[V(x)$ queries f at $w\|i]$ is small.
5. Soundness of ARG + $x \notin L_{f^\star} \Longrightarrow \Pr[V^{f^\star}(x, \pi_{f^\star}) = 1]$ is small for efficiently generated $\pi_{f^\star}$.
6. Point 4 $\Longrightarrow \forall w \in Q_{f^\star}, \Pr[V^{f_w}(x, \pi_{f^\star}) = 1] \approx \Pr[V^{f^\star}(x, \pi_{f^\star}) = 1]$.
7. Point 5 + 6 $\Longrightarrow \forall w \in Q_{f^\star}, \Pr[V^{f_w}(x, \pi_{f^\star}) = 1]$ is small, contradicting completeness of ARG.

Discussion and open problems

Low-degree random oracle model

Low-degree random oracle (LDROM) $\mathscr{P} := \{\mathscr{P}_\ell\}_{\ell \in \mathbb{N}}$

$\mathscr{P}_\ell$ is the uniform distribution over all polynomials $f\colon \mathbb{F}_{q(\ell)}^{n(\ell)} \to \mathbb{F}_{q(\ell)}$ of individual degree at most $d(\ell)$.

Open problem 1. Rule out relativized SNARGs in the LDROM, secure against query-bounded adversaries.

Do they exist or not??

[CCS22] construction:
Relativized SNARGs in the LDROM
secure against query-bounded and time-bounded adversaries

Our proof:
Can't generalize, no guarantee that $f_w \in \mathscr{P}[q,d]$.

	$w\|1$	$w\|2$	$w\|3$	$w\|4$	$w\|5$
$f^\star$	1001	0111	1110	0000	1010
f_w	0001	0111	0110	0000	0010

[CL20] impossibility:
No relativized PCPs in the LDROM
(PCPs are common subroutines in SNARGs constructions)
Caveat: only proved it for specific $f \in \mathscr{P}[q,d]$,
instead of a uniformly sampled $f \leftarrow \mathscr{P}[q,d]$

Characterization

Easy to learn/predict — Structured Oracle — Easy to construct relativized SNARGs: Learn the oracle and use non-relativized SNARGs

Low-Degree Random Oracle — Possible to construct relativized SNARGs secure against query-bounded and time-bounded adversaries

Conjecture: no relativized SNARGs secure against query-bounded adversaries

Hard to learn/predict — Random Oracle — No relativized SNARGs

Open problem 2.

Give a sufficient and necessary condition for an oracle that separates $\mathsf{DTIME}/\mathsf{NTIME}$ and relativized arguments.

Insights into Fiat-Shamir

Interactive protocol in the standard model → (Fiat-Shamir transformation) → Non-Interactive protocol in the ROM → (Heuristic instantiation) → Non-Interactive protocol in the standard model

SOMETIMES INSECURE!
Diagonalization attacks: [GK03;CGH04;BBHMR19;KRS25]

Relativized interactive protocol in the ROM → (XFS [AY25]) → Relativized non-interactive protocol In the ROM

Proven secure in the ROM
Natural class of white-box attacks "relativize"
(FS[relativized protocol] is insecure in the ROM)
$\Longrightarrow$ XFS is secure against many existing attacks

Is Fiat-Shamir transformation secure in other oracle models? LDROM? AROM?

Thank you!

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