Depth-3 Circuits for Inner Products

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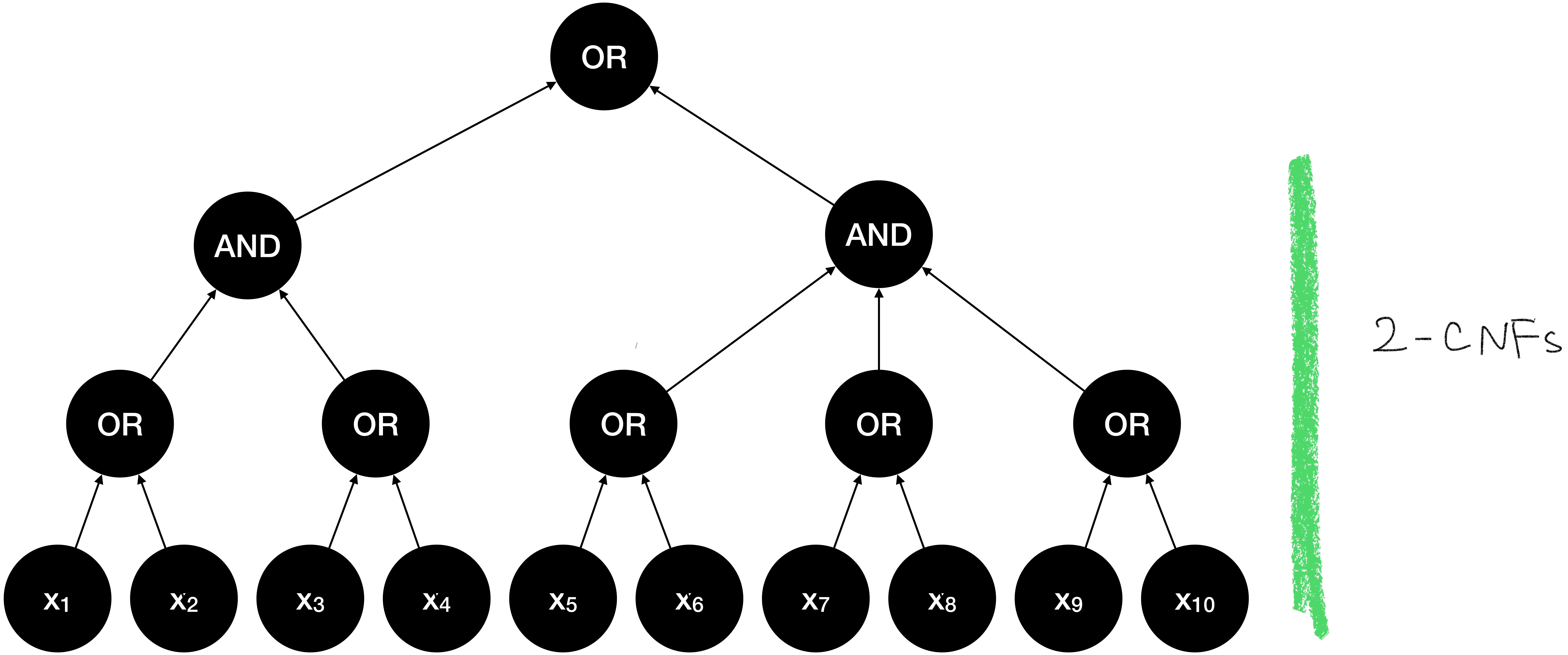
Σ_3^2: depth-3 bottom fan-in 2 circuits



Inner Product

- $\mathrm{IP}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$
- $\mathrm{IP}_n(x, y) := \langle x, y \rangle \bmod 2$
- $\mathrm{IP}_9(x, y) = 1$

1	0	1	0	0	1	0	0	1	$x \in \{0,1\}^9$
0	0	1	0	1	1	1	0	1	$y \in \{0,1\}^9$

Why Σ_3^2? Why Inner-Product?

- Σ_3^3-complexity of IP_n was raised as an open problem by Golovnev, Kulikov, and Williams
- Frankl, Gryaznov, and Talebanfard pointed out the challenge for obtaining Σ_3^2 -complexity of IP_n
- New techniques from related areas

Our Result

- Σ_3^2-complexity of IP_n is $2^{\alpha_n \cdot n}$ where $0.847\ldots \leq \alpha_n \leq 0.965\ldots$
- Previously, known: $\frac{1}{2} \leq \alpha_n \leq 1$

Upper Bound: $\alpha_n \leq 0.965\ldots$

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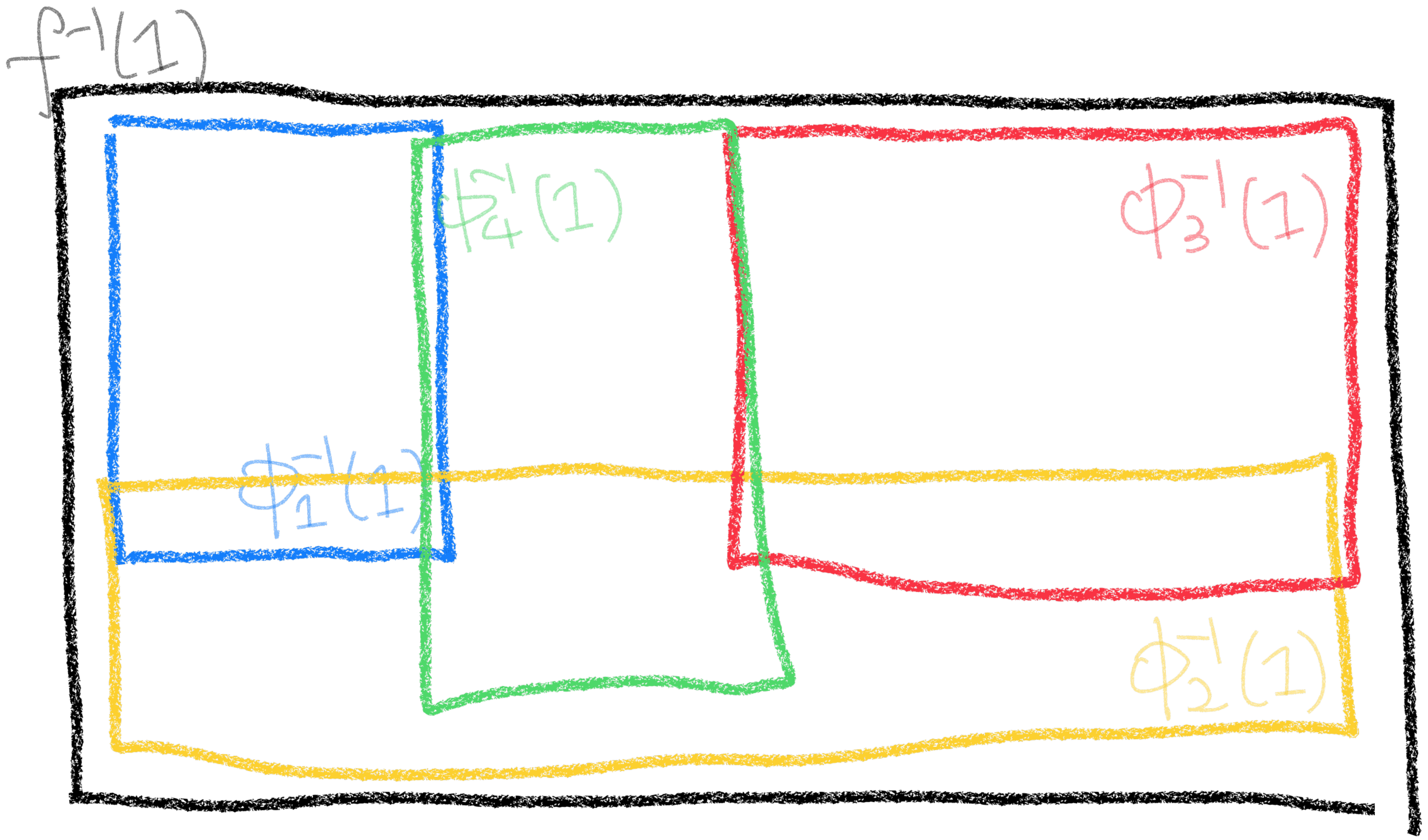
Fractional Cover [1/4]

- $f : \{0,1\}^n \to \{0,1\}$
- C : Σ_3^2-circuit with top fan-in m computing f
- $f^{-1}(1) = \cup_{i\in[m]} \phi_i^{-1}(1)$
  - ϕ_i: 2-CNF induced by C

Upper Bound: $\alpha_n \leq 0.965\ldots$

Fractional Cover [2/4]



Upper Bound: $\alpha_n \leq 0.965\ldots$

Fractional Cover [3/4]

- $\Phi := \{\phi : \text{2CNF}$ s.t. $\forall x \in \{0,1\}^n, \phi(x) \leq f(x)\}$
- Goal: Find minimum $S \subseteq \Phi$ where $f^{-1}(1) \subseteq \cup_{\phi \in S} \phi^{-1}(1)$
  - Σ_3^2-complexity of $f \leq O(|S|)$

Upper Bound: $\alpha_n \leq 0.965\ldots$

Fractional Cover [4/4]

$$\min \sum_{\phi \in \Phi} w_\phi$$

subject to $$\sum_{\phi \in \Phi} w_\phi \phi(x) \geq 1, \forall x \in f^{-1}(1)$$

$$w_\phi \in [0,1], \forall \phi \in \Phi$$

Note: The least top fan-in is captured by the corresponding integer program

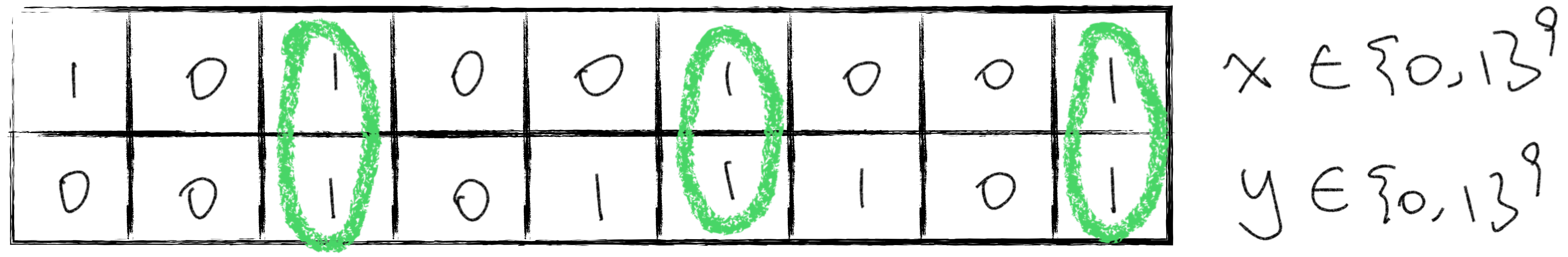
Lovasz's fractional cover

$w_\phi \in \{0, 1\}$

$\Rightarrow Opt \leq Opt^{\mathbb{Z}} \leq O(n) \cdot Opt$

Upper Bound: $\alpha_n \leq 0.965\ldots$

Construct Circuit [1/2]

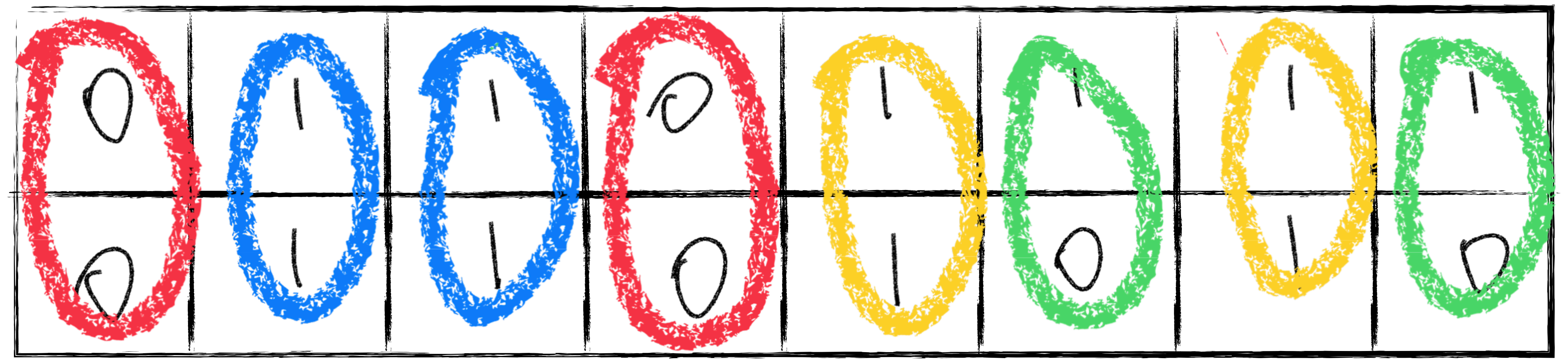


- $\mathrm{IP}_n(x, y)$ is determined by the parity of “11” patterns
- $p_{11}(x, y) := \frac{1}{n} \left| \{i \in [n] : (x_i, y_i) = (1,1)\} \right|$
- Case on p_{11} small and p_{11} large

Upper Bound: $\alpha_n \leq 0.965\ldots$

Construct Circuit [2/2]

- p_{11} small
  - Hardcode all $(x, y) \in \mathrm{IP}_n^{-1}(1)$ into a relatively small circuit
- p_{11} large
  - Handle $(x, y) \in \mathrm{IP}_n^{-1}(0)$ by cancelling out matching pairs: use LP



Lower Bound: $\alpha_n \geq 0.847\ldots$

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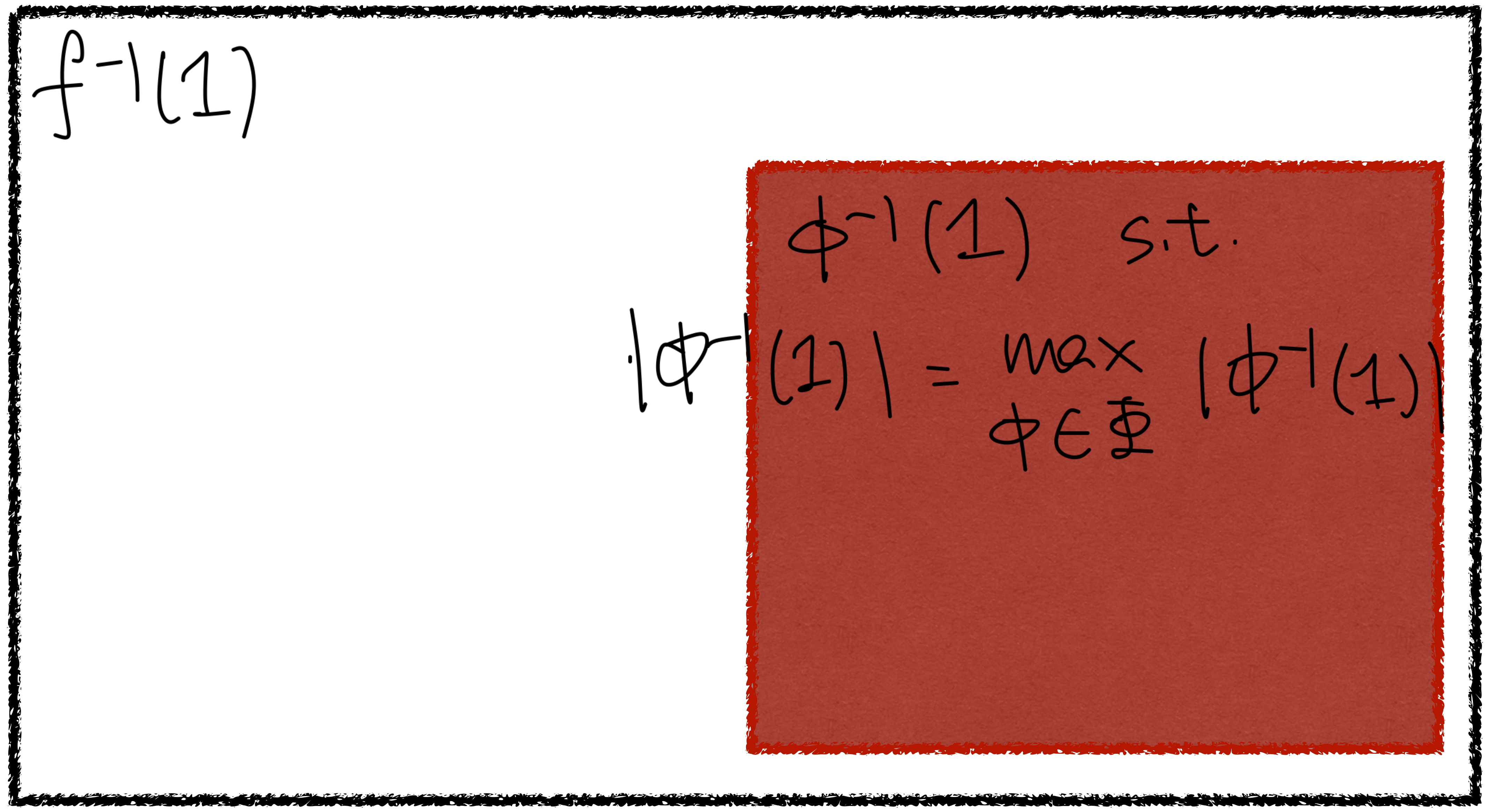
Fractional Cover [1/3]

- $\max_{\phi \in \Phi} \left| \phi^{-1}(1) \right| \leq M \implies$ top fan-in of $f \geq \dfrac{f^{-1}(1)}{M}$



Lower Bounds: $\alpha_n \geq 0.847\ldots$

Fractional Cover [2/3]

- Choose a hard distribution $\mathscr{D}$ over $f^{-1}(1)$
- Measure $\left|\phi^{-1}(1)\right|$ relative to $\mathscr{D}$
- $\max_{\phi \in \Phi} \Pr_{x \in \mathscr{D}}[\phi(x) = 1] \leq p \implies$ top fan-in of $f \geq \dfrac{1}{p}$

Lower Bounds: $\alpha_n \geq 0.847\ldots$

Fractional Cover [3/3]

- Hard distribution $\mathscr{D}_p$ over $\mathrm{IP}_n^{-1}(1)$: Binomial distribution where

$$(X, Y) \sim \mathscr{D}_p : \Pr[X_i = 1] = \Pr[Y_i = 1] = p$$

- Max-formulas

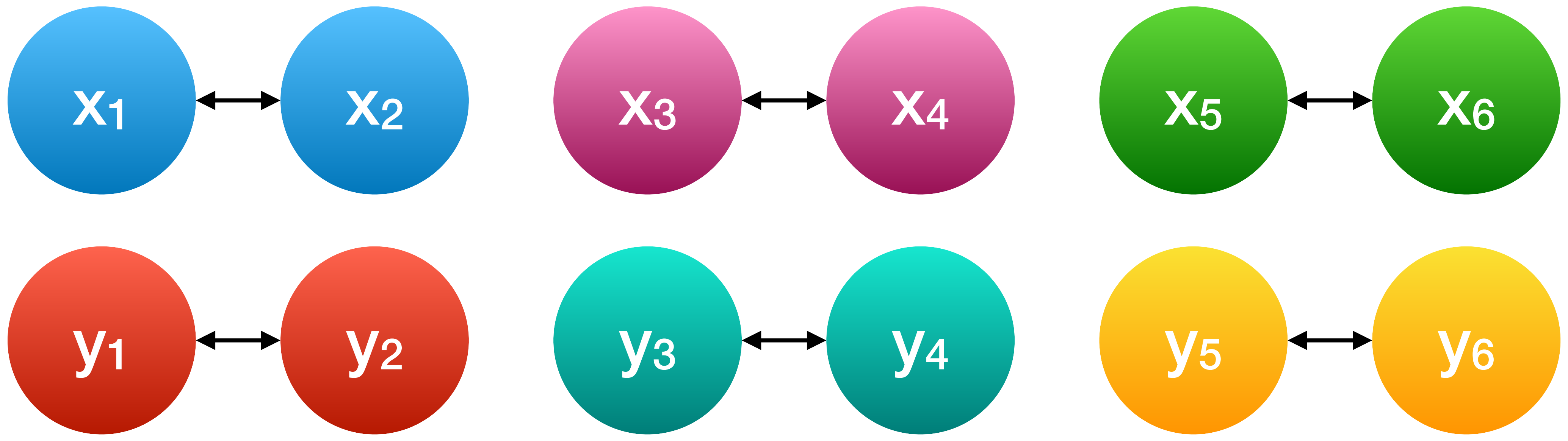
$$\forall \phi \in \Phi, \Pr_{(x,y)\sim\mathscr{D}_p}[\phi(x, y) = 1] \leq P_M$$

where P_M is the acceptance probability of the max-formulas

- Σ_3^2-complexity of $\mathrm{IP}_n \geq \dfrac{1}{P_M}$

Lower Bounds: $\alpha_n \geq 0.847\ldots$
Max Formulas [1/2]

- $\phi^{(n)}_{\text{Coll}} := \bigwedge_{i \in [n/2]} (x_{2i-1} \leftrightarrow x_{2i}) \wedge (y_{2i-1} \leftrightarrow y_{2i})$
- $\phi^{(n)}_{Nand} := \bigwedge_{i \in [n]} (\neg x_i \vee \neg y_i)$



Lower Bounds: $\alpha_n \geq 0.847\ldots$

Max Formulas [2/2]

- Structure lemma:
  - There exists a finite set S_n of formulas where

$$\forall \phi \in \Phi, \exists \phi' \in S_n \text{ s.t. } \phi \vDash \phi' \; (\phi^{-1}(1) \subseteq (\phi')^{-1}(1))$$

- There exists $p^* \in [0,1]$ where

$$\forall \phi \in S_n, \Pr_{\mathscr{D}_{p^*}}[\phi] \leq \max\{\Pr_{\mathscr{D}_{p^*}}[\phi_{Coll}], \Pr_{\mathscr{D}_{p^*}}[\phi_{Nand}]\}$$

https://eccc.weizmann.ac.il/report/2023/106/